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A Revenue Equivalence Result in a Duopolistic Electricity Market where one of the suppliers has two production units

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Abstract

In this paper we will model the electricity market auction as a two-person game with incomplete information under the assumption that bidders are asymmetric in units production, risk neutral and with unknown values.

We characterize the strictly monotone bayesian Nash equilibrium and we rank a family of auction models which contains the classic models Uniform, Discriminatory and Vickrey auction models.

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1 Introduction

Gradually since the 1990s, several countries have liberalized their electricity markets, initially in state hands. In these countries, power companies compete to generate in the Electricity Market and they take their electricity production to auction [Fehr y Harbord(1998)]. Each power company bids an amount of electricity units and a unit price for each hour (or half hour) of the following day. In view of the supply, the Market Operator (the auctioneer) ranks the bids from the lowest to the highest and then distributes the demand among the lowest bids, until the demand has been fully met. The price paid to each company taking part in the dispatch of the demand, depends on the auction model adopted for the transaction. There are two main auction models: the Uniform auction model and the Discriminatory auction model.

In the Uniform auction model, the unit price received by a company supplying the market is the same for all companies: the highest accepted bid. In the Discriminatory auction model each company dispatching in the market receives its own bid. Much debate has been going on about the advantages and disadvantages of these auction models [Ausubel and Cramton (2002), Fabra (2001); Fabra, Fehr and Harbord (2002); Fabra, Fehr and Harbord (2003)] but no clear conclusion has been reached.

Some papers argue in favour of the Uniform model [Wolfram (1999)] whilst others favour the Discriminatory model [Federico and Rahman (2001)]. There are other auction models used in contexts outside the Electricity Market such as the Vickrey auction model [Vickrey (1961)].

This paper will be consider a parametric family of auction models which contains the three classic models mentioned. We will make a comparative analysis of the auction models belonging to the parametric family, identifying the preferred model of the companies or the Market Operator.

In this paper the valuations are private, so we model the Electricity Market as a game with incomplete information.

The organization and the achievements of this work are:

We define a parametric family of auction models, which contains as particular cases, the Uniform, the Discriminatory and the Vickrey auction models. We define the hypotheses in Section 2, which includes that one of the suppliers has two production units.

In Section 3, we obtain bayesian Nash equilibria for every auction model belonging to the parametric family. Once the equilibria have been determined, we calculate the expected revenue for the companies and the payment the Market Operator expects to make in Section 4. These expressions are not dependent on the choice of auction model belonging to the vertices of $\mathcal{GAM}$ and we obtain a revenue equivalence result.

2 The Market Model

We assume the following hypotheses in the market model:

There are two risk neutral suppliers where one of them, called supplier 1, has two production units denoted by subindexes $\widetilde{1}$, $\widehat{1}$. The other supplier, called supplier 2, has only one production unit, denoted by subindex 2. Both suppliers compete to provide the electricity required and they have the same perfectly divisible capacity $k_{\widetilde{1}} = k_{\widehat{1}} = 1$ and $k_2 = 2$.

The cost function of production unit $i \in \{\widetilde{1}, \widehat{1}, 2\}$ is $g(q_i, \theta_i) = q_i\theta_i$, where $q_i \in [0, 1]$ is the amount dispatched by production unit i.

The type θ_i $\left(\text{where } \theta_{\widetilde{1}} = \theta_{\widehat{1}} = \theta_1 \text{ and } \theta_2\right)$ which is private information to supplier $i \in \{1, 2\}$, is an independent realization of a uniformly distributed continuous random variable in $[0,1]$. Supplier i and only supplier i, observes the realization of θ_i and it gathers the uncertainty that company j has about the production cost of company i.

The demand of a period is price-inelastic, known with certainty and is represented by the parameter D.

Each supplier simultaneously and independently submits a bid: $\left(b_{\tilde{1}}, b_{\hat{1}}\right) \in [0,1] \times [0,1]$ is the bid of supplier 1 (one component by each production unit) and $b_2 \in [0,1]$ is the bid of supplier 2, specifying the minimum unit-price offer at which it is willing to supply the whole of the capacity of each production unit.

A strategy for production unit $i \in \{\widetilde{1}, \widehat{1}, 2\}$ is a strictly monotone and differentiable function $b_i(.) : [0,1] \times [0,1]$.

Let's refer to this market model as the **Asymmetric Model in production units**.

Once the Market Operator has received the bids, it allocates the electricity distribution in such a manner that the production unit with the lowest bid will dispatch first. If its capacity is not enough to satisfy the entire demand, then the production unit with the second lowest bid will dispatch second. If its capacities are not enough to satisfy the entire demand, then the production unit with the highest bid will satisfy the residual demand. Hence the amount that the supplier $i \in \{1,2\}$ dispatches is given by the following function:

$$Q_1(b_{\tilde{1}}, b_{\hat{1}}, b_2) = \begin{cases} \min(2, D) & \text{if} \quad M < b_2 \\ \min(1, D) & \text{if} \quad m < b_2 < M \text{ and } D < 3 \\ 1 + \min(1, D-3) & \text{if} \quad m < b_2 < M \text{ and } 3 < D \\ 0 & \text{if} \quad b_2 < m \text{ and } D < 2 \\ \min(2, D-2) & \text{if} \quad b_2 < m \text{ and } 2 < D \end{cases}$$

$$Q_2(b_{\tilde{1}}, b_{\hat{1}}, b_2) = \begin{cases} \min(2, D) & \text{if} \quad b_2 < m \\ 0 & \text{if} \quad m < b_2 < M \text{ and } D < 1 \\ \min(2, D-1) & \text{if} \quad m < b_2 < M \text{ and } 1 < D \\ 0 & \text{if} \quad M < b_2 \text{ and } D < 2 \\ \min(2, D-2) & \text{if} \quad M < b_2 \text{ and } 2 < D \end{cases}$$

Where $m = \min(b_{\tilde{1}}, b_{\hat{1}})$ and $M = max(b_{\tilde{1}}, b_{\hat{1}})$.

All aspects of this game and the auction model used, are assumed to be common knowledge.

The price paid to each supplier depends on the auction model adopted for the transaction. There are three classic auction models: **Uniform auction model, Discriminatory auction model and Vickrey auction model**.

In the Uniform auction model, the unit-price received by production unit i is equal to the highest accepted bid. All production units in the market receive the same unit-price.

In the Discriminatory auction model, the unit-price received by production unit i is equal to its own bid b_i. All production units dispatching into the market could receive a different unit-price.

In the Vickrey auction model, the rule used by the Market Operator to establish the price is more complicated than in the previous two models. The unit price received by production unit i dispatching in the market is equal to the unit price of the electricity unit needed to cover the demand if the supplier who has the production unit i removes its bid.

This paper discusses not only the three classic models, but in fact we will also consider a parametric family of auction models which contains the three classic models as particular cases. This family is a set of auction models whose profit function for supplier i is:

$$B_1(\theta_1, b_{\widetilde{1}}, b_{\widehat{1}}, b_2) = \begin{cases} \gamma_1^{\widetilde{1}} m + \gamma_1^{\widehat{1}} M + \beta_1^1 b_2 + \varphi - \phi_1 \theta_1 & \text{if} \quad M < b_2 \\ \gamma_1^{\widetilde{1}} m + \gamma_2^{\widehat{1}} M + \beta_2^1 b_2 + \varphi - \phi_2^1 \theta_1 & \text{if} \quad m < b_2 < M \\ \gamma_3^{\widetilde{1}} m + \gamma_2^{\widehat{1}} M + \varphi - \phi_3 \theta_1 & \text{if} \quad b_2 < m \end{cases} \quad 2.1.a)$$

$$B_2(\theta_2, b_{\widetilde{1}}, b_{\widehat{1}}, b_2) = \begin{cases} \gamma_1^2 b_2 + \beta_1^{\widetilde{2}} m + \beta_1^{\widehat{2}} M + \varphi - \phi_1 \theta_2 & \text{if} \quad b_2 < m \\ \gamma_2^2 b_2 + \beta_1^{\widehat{2}} M + \varphi - \phi_2^2 \theta_2 & \text{if} \quad m < b_2 < M \\ \gamma_3^2 b_2 + \varphi - \phi_3 \theta_2 & \text{if} \quad M < b_2 \end{cases} \quad 2.1.b)$$

Where $m = \min(b_{\widetilde{1}}, b_{\widehat{1}})$ and $M = max(b_{\widetilde{1}}, b_{\widehat{1}})$

$$\gamma_1^{\widetilde{1}}, \gamma_1^{\widehat{1}}, \gamma_2^{\widehat{1}}, \gamma_3^{\widetilde{1}}, \gamma_1^2, \gamma_2^2, \gamma_3^2, \beta_1^1, \beta_2^1, \beta_1^{\widetilde{2}}, \beta_1^{\widehat{2}}, \varphi \in [0, \infty)$$

$$\gamma_1^{\widetilde{1}} + \gamma_1^{\widehat{1}} + \beta_1^1 + \varphi = \phi_1, \quad \gamma_1^{\widetilde{1}} + \gamma_2^{\widehat{1}} + \beta_2^1 + \varphi = \phi_2^1, \quad \gamma_3^{\widetilde{1}} + \gamma_2^{\widehat{1}} + \varphi = \phi_3$$

$$\gamma_1^2 + \beta_1^{\widetilde{2}} + \beta_1^{\widehat{2}} + \varphi = \phi_1, \quad \gamma_2^2 + \beta_1^{\widehat{2}} + \varphi = \phi_2^2, \quad \gamma_3^2 + \varphi = \phi_3, \quad \gamma_1^{\widetilde{1}} + \gamma_1^{\widehat{1}} = \gamma_1^2$$

$$\beta_1^{\widetilde{2}} + \beta_1^{\widetilde{2}} = \beta_1^1, \quad \gamma_3^{\widetilde{1}} + \gamma_2^{\widehat{1}} = \gamma_3^2$$

The parameters ϕ_1, ϕ_2^1, ϕ_2^2 and ϕ_3 are determined by the demand. If $D \leq 1$, then $\phi_1 = D$, $\phi_2^1 = \phi_2^2 = \phi_3 = 0$. If $D = 1 + \alpha$, with $\alpha \in (0, 1]$, then $\phi_1 = 1 + \alpha$, $\phi_2^1 = 1$, $\phi_2^2 = \alpha$, $\phi_3 = 0$. If $D = 2 + \alpha$, with $\alpha \in (0, 1]$, then $\phi_1 = 2$, $\phi_2^1 = 1$, $\phi_2^2 = 1 + \alpha$, $\phi_3 = \alpha$. If $D = 3 + \alpha$, with $\alpha \in (0, 1]$, then $\phi_1 = 2$, $\phi_2^1 = 1 + \alpha$, $\phi_2^2 = 2$, $\phi_3 = 1 + \alpha$. If $D \geq 4$, then $\phi_1 = 2$, $\phi_2^1 = 2$, $\phi_2^2 = 2$, $\phi_3 = 2$.

This family of auction models verifies two principles required for an electricity auction, which are:

- The bid made by a production unit is the minimum price at which it is willing to supply the whole of its capacity. The Market Operator cannot pay a company a price lower than its own bid.
- The production unit that has made the lowest bid should enter the market first and if it cannot satisfy all the demand, then the others production units should enter the market to dispatch the residual demand.

Let's refer to this parametric family of auction models as The **General Auction Model** ($\mathcal{GAM}$).

Clearly, if we fix the values of the parameters then the auction model used for the transaction is completely determined. The values of the parameters depend on the size of the demand. Therefore we will analyze the following cases:

Case 1 The three production units have enough capacity to supply the whole demand, i.e., $D \leq 1$. In this case, the production unit with the lowest bid is the only one to dispatch. In fact, Case 1 is a particular case of single-unit (D) auction. Therefore, we know beforehand that the Revenue Equivalence Theorem is applicable. The expected revenue for each supplier is $P_i(\theta_i) = D\frac{(1-\theta_i^2)}{2}$ and the Market Operator expects to pay $P_{MO} = \frac{2}{3}D$.

Case 2 Supplier 2 has enough capacity to supply the whole demand with its unique production unit. However the supplier 1, who has enough capacity too, needs its two production units for this, i.e. $1 < D = 1 + \alpha \leq 2$, with $\alpha \in (0,1)$. In this case if bidder 2 is the bidder with the lowest bid, then bidder 2 is the only one to dispatch. If bidder 2 puts its bid between the two bids of bidder 1 then: one of the production unit of supplier 1 dispatches all its capacity and bidder 2 dispatches the residual demand α. Finally, if the bid of bidder 2 is the highest of the three bids, then bidder 2 doesn't dispatch any electricity units.

Then, substituting $\phi_1 = 1 + \alpha$, $\phi_2^1 = 1$, $\phi_2^2 = \alpha$ and $\phi_3 = 0$ into the expressions 2.1.a) and 2.1.b), the $\mathcal{GAM}$ is reduced to a parametric family with profit function for supplier $i \in \{1,2\}$:

$$B_1(\theta_1, b_{\tilde{1}}, b_{\hat{1}}, b_2) =$$

$$= \begin{cases} \gamma_1^{\tilde{1}} m + \left(\gamma_1^2 - \gamma_1^{\tilde{1}}\right) M + (1+\alpha-\gamma_1^2) b_2 - (1+\alpha)\,\theta_1 & \text{if} \quad M < b_2 \\ \gamma_1^{\tilde{1}} m + \left(1 - \gamma_1^{\tilde{1}}\right) b_2 - \theta_1 & \text{if} \quad m < b_2 < M \\ 0 & \text{if} \quad b_2 < m \end{cases}$$

$$B_2(\theta_2, b_{\tilde{1}}, b_{\hat{1}}, b_2) =$$

$$= \begin{cases} \gamma_1^2 b_2 + \left(1+\alpha-\beta_1^{\hat{2}} - \gamma_1^2\right) m + \beta_1^{\hat{2}} M - (1+\alpha)\,\theta_2 & \text{if} \quad b_2 < m \\ \left(\alpha - \beta_1^{\hat{2}}\right) b_2 + \beta_1^{\hat{2}} M - \alpha\theta_2 & \text{if} \quad m < b_2 < M \\ 0 & \text{if} \quad M < b_2 \end{cases}$$

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where $\left(\gamma_1^{\widetilde{1}}, \gamma_1^2, \beta_1^{\widehat{2}}\right) \in [0,1] \times \left[\gamma_1^{\widetilde{1}}, 1+\alpha\right] \times \left[0, \min\{\alpha, 1+\alpha-\gamma_1^2\}\right]$, $m = \min(b_{\widetilde{1}}, b_{\widehat{1}})$ and $M = max(b_{\widetilde{1}}, b_{\widehat{1}})$.

Case 3 The capacity of both suppliers is needed to satisfy the demand, but possibly the three production units wouldn't be necessary, i.e., $2 < D = 2+\alpha < 3$, with $\alpha \in (0,1)$. In this case, if supplier 2 has the highest bid then the three production units enter the market: the two production units of supplier 1 despatch all its capacity and supplier 2 despatches the residual demand α. Otherwise, if supplier 2 doesn't have the highest bid, the production unit of supplier 1 with the highest bid does not enter the market.

Then, substituting $\phi_1 = 2$, $\phi_2^1 = 1$, $\phi_2^2 = 1+\alpha$ and $\phi_3 = \alpha$ into the expressions 2.1.a) and 2.1.b), the $\mathcal{GAM}$ is reduced to a parametric family with profit function for supplier $i \in \{1,2\}$:

$$B_1(\theta_1, b_{\widetilde{1}}, b_{\widehat{1}}, b_2) =$$

$$= \begin{cases} \gamma_1^{\widetilde{1}} m + \left(\gamma_1^2 - \gamma_1^{\widetilde{1}}\right) M + \left(2-\alpha+\gamma_3^2-\gamma_1^2\right) b_2 + \alpha - \gamma_3^2 - 2\theta_1 & \text{if} \quad M < b_2 \\ \gamma_1^{\widetilde{1}} m + \gamma_2^{\widehat{1}} M + \left(1-\alpha+\gamma_3^2-\gamma_2^{\widehat{1}}-\gamma_1^{\widetilde{1}}\right) b_2 + \alpha - \gamma_3^2 - \theta_1 & \text{if} \quad m < b_2 < M \\ \left(\gamma_3^2 - \gamma_2^{\widehat{1}}\right) m + \gamma_2^{\widehat{1}} M + \alpha - \gamma_3^2 - \alpha\theta_1 & \text{if} \quad b_2 < m \end{cases}$$

$$B_2(\theta_2, b_{\widetilde{1}}, b_{\widehat{1}}, b_2) =$$

$$= \begin{cases} \gamma_1^2 b_2 + \left(2-\alpha+\gamma_3^2-\beta_1^{\widehat{2}}-\gamma_1^2\right) m + \beta_1^{\widehat{2}} M + \alpha - \gamma_3^2 - 2\theta_2 & \text{if} \quad b_2 < m \\ \left(1+\gamma_3^2-\beta_1^{\widehat{2}}\right) b_2 + \beta_1^{\widehat{2}} M + \alpha - \gamma_3^2 - (1+\alpha)\,\theta_2 & \text{if} \quad m < b_2 < M \\ \gamma_3^2 b_2 + \alpha - \gamma_3^2 - \alpha\theta_2 & \text{if} \quad M < b_2 \end{cases}$$

where

$$\left(\gamma_3^2, \gamma_2^{\widehat{1}}, \gamma_1^{\widetilde{1}}, \gamma_1^2, \beta_1^{\widehat{2}}\right) \in [0,\alpha] \times \left[0, \gamma_3^2\right] \times \left[0, 1-\alpha+\gamma_3^2-\gamma_2^{\widehat{1}}\right] \times$$
$$\left[\gamma_1^{\widetilde{1}}, 2-\alpha+\gamma_3^2\right] \times \left[0, \min\{2-\alpha+\gamma_3^2-\gamma_1^2, 1+\gamma_3^2\}\right]$$

$m = \min(b_{\widetilde{1}}, b_{\widehat{1}})$ and $M = max(b_{\widetilde{1}}, b_{\widehat{1}})$.

Case 4 The capacity of three production units suppliers is needed to satisfy the demand but there is excess overall capacity, i.e. $3 \leq D = 3+\alpha < 4$, with $\alpha \in (0,1)$. In this case, one production unit of supplier 1 despatches all its capacity and the other production unit of supplier 1 despatches at least the residual demand α. If supplier 2 doesn't have the highest bid, then it despatches all its capacity and otherwise it despatches $1+\alpha$.

Then, substituting $\phi_1 = 2$, $\phi_2^1 = 1+\alpha$, $\phi_2^2 = 2$ and $\phi_3 = 1+\alpha$ into the expressions 2.1.a) and 2.1.b), the $\mathcal{GAM}$ is reduced to a parametric family with profit function for supplier $i \in \{1,2\}$:

$B_1(\theta_1, b_{\tilde{1}}, b_{\hat{1}}, b_2) =$

$$= \begin{cases} \gamma_1^{\tilde{1}} m + \left(\gamma_1^2 - \gamma_1^{\tilde{1}}\right) M + (1 - \alpha + \gamma_3^2 - \gamma_1^2)\, b_2 + 1 + \alpha - \gamma_3^2 - 2\theta_1 & \text{if } M < b_2 \\ \gamma_1^{\tilde{1}} m + \gamma_2^{\hat{1}} M + \left(\gamma_3^2 - \gamma_1^{\tilde{1}} - \gamma_2^{\hat{1}}\right) b_2 + 1 + \alpha - \gamma_3^2 - (1+\alpha)\,\theta_1 & \text{if } m < b_2 < M \\ \left(\gamma_3^2 - \gamma_2^{\hat{1}}\right) m + \gamma_2^{\hat{1}} M + 1 + \alpha - \gamma_3^2 - (1+\alpha)\,\theta_1 & \text{if } b_2 < m \end{cases}$$

$B_2(\theta_2, b_{\tilde{1}}, b_{\hat{1}}, b_2) =$

$$\begin{cases} \gamma_1^2 b_2 + \left(1 - \alpha + \gamma_3^2 - \beta_1^{\hat{2}} - \gamma_1^2\right) m + \beta_1^{\hat{2}} M + 1 + \alpha - \gamma_3^2 - 2\theta_2 & \text{if} \quad b_2 < m \\ \left(1 - \alpha + \gamma_3^2 - \beta_1^{\hat{2}}\right) b_2 + \beta_1^{\hat{2}} M + 1 + \alpha - \gamma_3^2 - 2\theta_2 & \text{if} \quad m < b_2 < M \\ \gamma_3^2 b_2 + 1 + \alpha - \gamma_3^2 - (1+\alpha)\,\theta_2 & \text{if} \quad M < b_2 \end{cases}$$

where

$$\left(\gamma_3^2, \gamma_2^{\hat{1}}, \gamma_1^{\tilde{1}}, \gamma_1^2, \beta_1^{\hat{2}}\right) \in [0, 1+\alpha] \times \left[0, \gamma_3^2\right] \times \left[0, \gamma_3^2 - \gamma_2^{\hat{1}}\right] \times \left[\gamma_1^{\tilde{1}}, 1 - \alpha + \gamma_3^2\right] \times \left[0, 1 - \alpha + \gamma_3^2 - \gamma_1^2\right]$$

$m = \min(b_{\tilde{1}}, b_{\hat{1}})$ and $M = max(b_{\tilde{1}}, b_{\hat{1}})$.

Case 5 Demand exceeds overall capacity, i.e. $4 \leq D$. In this case there is no competition. Both companies are guaranteed to dispatch their entire capacity. The revenue for each supplier is 2 and the Market Operator pay 4. Obviously this is a trivial case.

3 Bayesian Nash Equilibrium

The following proposition gives the bayesian Nash equilibria for any auction model belonging to the General Auction Model.

Proposition 1. *If the hypotheses of the* ***Asymmetric Model in production units*** *hold and an auction model belonging to $\mathcal{GAM}$ is used, then:*

Region I) *If $\phi_3 = \phi_2^1$, $\phi_1 = \phi_2^2$, $\gamma_1^{\tilde{1}} = 0$, $\gamma_2^{\hat{1}} = \gamma_3^2$, $\phi_1 - \phi_3 + \gamma_3^2 - \beta_1^{\hat{2}} - \gamma_1^2 = 0$ then there exist infinite bayesian Nash equilibria are given by*

$$\left(b_{\tilde{1}}^*(\theta_1), b_{\hat{1}}^*(\theta_1), b_2^*(\theta_2)\right) = \left(b_{\tilde{1}}^*(\theta_1), b^*(\theta_1), b^*(\theta_2)\right)$$

where $b_{\tilde{1}}^*(\theta_1)$ is any strictly monotone and differentiable function verifying $b_{\tilde{1}}^*(\theta_1) \leq b^*(\theta_1)$ $\forall \theta_1 \in [0,1]$.

Region II) *Otherwise if one of the following expressions is true*

•$\gamma_2^{\hat{1}} \neq 0$ *or* $\gamma_3^2 \neq 0$

•$\gamma_2^{\hat{1}} = 0$ *and* $\gamma_1^{\tilde{1}} = \gamma_1^2$

•$\gamma_2^{\hat{1}} = \gamma_3^2 = 0$ *and* $\gamma_1^2 = (1+k)\,\gamma_1^{\tilde{1}}$, $(1+k)\,\phi_2^1 = \phi_1 + k\phi_3$, *for any* $k \in \Re$

then the unique symmetric bayesian Nash equilibrium is given by:

$$\left(b_{\tilde{1}}^{*}\left(\theta_{1}\right), b_{\hat{1}}^{*}\left(\theta_{1}\right), b_{2}^{*}\left(\theta_{2}\right)\right)=\left(b^{*}(\theta_{1}), b^{*}(\theta_{1}), b^{*}(\theta_{2})\right)$$

*where $b^*(\theta_i)$ in **Region I** and **Region II** is given by:*

a) If $\gamma_1^2 = \gamma_3^2 = 0$ then $b^*(\theta_i) = \theta_i$

b) If $\gamma_3^2 = 0$ and $\gamma_1^2 \neq 0$ then

$$b^*(\theta_i) = \frac{(\phi_1 - \phi_3)\,\theta_i + \gamma_1^2}{\gamma_1^2 + \phi_1 - \phi_3}$$

c) If $\gamma_1^2 \neq \gamma_3^2 \neq 0$ then

$$b^*(\theta_i) = \frac{(\phi_3 - \phi_1)\,\theta_i - \gamma_1^2 + \left(\gamma_3^2\right)^{\frac{\gamma_3^2 - \gamma_1^2 + \phi_3 - \phi_1}{\gamma_3^2 - \gamma_1^2}} \left(\gamma_1^2 + \left(\gamma_3^2 - \gamma_1^2\right)\theta_i\right)^{\frac{\phi_1 - \phi_3}{\gamma_3^2 - \gamma_1^2}}}{\gamma_3^2 - \gamma_1^2 + \phi_3 - \phi_1}$$

d) If $\gamma_1^2 = \gamma_3^2 \neq 0$ then

$$b^*(\theta_i) = \theta_i + \frac{\gamma_1^2\left(1 - e^{\frac{-(1-\theta_i)(\phi_1 - \phi_3)}{\gamma_1^2}}\right)}{\phi_1 - \phi_3}$$

***Region III)** If the parameters don´t belong to **Region I** or **Region II** then the bayesian Nash equilibria are solutions of the following system of differential equations:*

$$\begin{cases} \gamma_1^{\tilde{1}}\left(1-\left(b_2^*\right)^{-1}(t)\right)+\left(\phi_3-\phi_2^1\right)\left(t-\left(b_{\tilde{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}\left(b_2^*\right)^{-1}(t)=0 \\ \left(\gamma_1^2-\gamma_1^{\tilde{1}}\right)\left(1-\left(b_2^*\right)^{-1}(t)\right)+\left(\phi_2^1-\phi_1\right)\left(t-\left(b_{\hat{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}\left(b_2^*\right)^{-1}(t)=0 \\ \gamma_1^2\left(1-(b_{\tilde{1}}^*)^{-1}(t)\right)+\left(\phi_2^2-\phi_3-\beta_1^{\hat{2}}\right)\left((b_{\tilde{1}}^*)^{-1}(t)-(b_{\hat{1}}^*)^{-1}(t)\right) \\ +\left(\phi_2^2-\phi_1\right)\left(t-\left(b_2^*\right)^{-1}(t)\right)\frac{d}{dt}(b_{\tilde{1}}^*)^{-1}(t) \\ +\left(\phi_3-\phi_2^2\right)\left(t-\left(b_2^*\right)^{-1}(t)\right)\frac{d}{dt}(b_{\hat{1}}^*)^{-1}(t)=0 \end{cases}$$

Proof. See appendix ☐

We will deduce the Bayesian Nash equilibrium for each non trivial case from Proposition 1:

3.1 Case 2

In this case, the parametric family of auction models is determined by the values of the parameters belonging to the region $RC2 = \{\left(\gamma_1^{\tilde{1}}, \gamma_1^2, \beta_1^{\hat{2}}\right) \in [0,1] \times$

$\left[\gamma_1^{\tilde{1}}, 1+\alpha\right] \times \left[0, \min\{\alpha, 1+\alpha-\gamma_1^2\}\right]\}$. There are eight auction models on the vertices of this region

Model	$\gamma_1^{\tilde{1}}$	γ_1^2	$\beta_1^{\hat{2}}$
A	0	0	0
Vickrey	0	0	α
Uniform	0	$1+\alpha$	0
B	1	1	0
C	1	1	α
Discriminatory	1	$1+\alpha$	0

with the following bayesian Nash equilibria

Model/Equilibrium	$\left[b_{\tilde{1}}^*(\theta_1), b_{\hat{1}}^*(\theta_1), b_2^*(\theta_2)\right]$
A	$[\theta_1, \theta_1, \theta_2]$
Vickrey	$[\theta_1, \theta_1, \theta_2]$
Uniform	$\left[\frac{1+\theta_1}{2}, \frac{1+\theta_1}{2}, \frac{1+\theta_2}{2}\right]$
B	$\left[\frac{1+(1+\alpha)\theta_1}{2+\alpha}, \frac{1+(1+\alpha)\theta_1}{2+\alpha}, \frac{1+(1+\alpha)\theta_2}{2+\alpha}\right]$
C	$\left[\frac{1+(1+\alpha)\theta_1}{2+\alpha}, \frac{1+(1+\alpha)\theta_1}{2+\alpha}, \frac{1+(1+\alpha)\theta_2}{2+\alpha}\right]$
Discriminatory	$\left[\frac{1+\theta_1}{2}, \frac{1+\theta_1}{2}, \frac{1+\theta_2}{2}\right]$

3.2 Case 3

In this case the parametric family of auction models is determined by the values of the parameters belonging to the region

$$RC3 = \left(\gamma_3^2, \gamma_2^{\hat{1}}, \gamma_1^{\tilde{1}}, \gamma_1^2, \beta_1^{\hat{2}}\right) \in [0,\alpha] \times \left[0, \gamma_3^2\right] \times \left[0, 1-\alpha+\gamma_3^2-\gamma_2^{\hat{1}}\right] \times \left[\gamma_1^{\tilde{1}}, 2-\alpha+\gamma_3^2\right] \times \left[0, \min\{2-\alpha+\gamma_3^2-\gamma_1^2, 1+\gamma_3^2\}\right]$$

. There are eighteen auction models on the vertices of this region:

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Model	γ_3^2	$\gamma_2^{\widehat{1}}$	$\gamma_1^{\widetilde{1}}$	γ_1^2	$\beta_1^{\widehat{2}}$
$A1$	0	0	0	0	0
Uniform	α	0	0	0	0
$A2$	α	α	0	0	0
$B1$	0	0	$1-\alpha$	$1-\alpha$	0
$B2$	α	0	1	1	0
$B3$	α	α	$1-\alpha$	$1-\alpha$	0
$C1$	0	0	$1-\alpha$	$1-\alpha$	1
$C2$	α	0	1	1	1
$C3$	α	α	$1-\alpha$	$1-\alpha$	$1+\alpha$
$D1$	0	0	$1-\alpha$	$2-\alpha$	0
Discriminatory	α	0	1	2	0
$D2$	α	α	$1-\alpha$	2	0
Vickrey	0	0	0	0	1
$V1$	α	0	0	0	$1+\alpha$
$V2$	α	α	0	0	$1+\alpha$
$U1$	0	0	0	$2-\alpha$	0
$U2$	α	0	0	2	0
$U3$	α	α	0	2	0

with the following bayesian Nash equilibria

Model/Equilibrium	$\left[b^*_{\tilde{1}}(\theta_1), b^*_{\hat{1}}(\theta_1), b^*_2(\theta_2)\right]$
$A1$	$[\theta_1, \theta_1, \theta_2]$
Uniform	$\left[\frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_2-\alpha\theta_2^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}\right]$
$A2$	$\left[\frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_2-\alpha\theta_2^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}\right]$
$B1$	$\left[\frac{(2-\alpha)\theta_1+1-\alpha}{3-2\alpha}, \frac{(2-\alpha)\theta_1+1-\alpha}{3-2\alpha}, \frac{(2-\alpha)\theta_2+1-\alpha}{3-2\alpha}\right]$
$B2$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(2-\alpha)\theta_i+1-\alpha^{\frac{3-2\alpha}{1-\alpha}}(1-(1-\alpha)\theta_i)^{\frac{\alpha-2}{1-\alpha}}}{3-2\alpha}$
$B3$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(2-\alpha)\theta_i+1-\alpha-\alpha^{\frac{3\alpha-3}{2\alpha-1}}(1-\alpha-(1-2\alpha)\theta_i)^{\frac{2-\alpha}{2\alpha-1}}}{3-3\alpha}$
$C1$	$\left[\frac{(2-\alpha)\theta_1+1-\alpha}{3-2\alpha}, \frac{(2-\alpha)\theta_1+1-\alpha}{3-2\alpha}, \frac{(2-\alpha)\theta_2+1-\alpha}{3-2\alpha}\right]$
$C2$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(2-\alpha)\theta_i+1-\alpha^{\frac{3-2\alpha}{1-\alpha}}(1-(1-\alpha)\theta_i)^{\frac{\alpha-2}{1-\alpha}}}{3-2\alpha}$
$C3$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(2-\alpha)\theta_i+1-\alpha-\alpha^{\frac{3\alpha-3}{2\alpha-1}}(1-\alpha-(1-2\alpha)\theta_i)^{\frac{2-\alpha}{2\alpha-1}}}{3-3\alpha}$
$D1$	$\left[\frac{1+\theta_1}{2}, \frac{1+\theta_1}{2}, \frac{1+\theta_2}{2}\right]$
Discriminatory	$\left[\frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_2^2}{2(2-(2-\alpha)\theta_2)}\right]$
$D2$	$\left[\frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_2^2}{2(2-(2-\alpha)\theta_2)}\right]$
Vickrey	$[\theta_1, \theta_1, \theta_2]$
$V1$	$\left[\frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_2-\alpha\theta_2^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}\right]$
$V2$	$\left[\frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_1-\alpha\theta_1^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}, \frac{(2-\alpha)\theta_2-\alpha\theta_2^{\frac{2-\alpha}{\alpha}}}{2(1-\alpha)}\right]$

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$U1$	$\left[\frac{1+\theta_1}{2}, \frac{1+\theta_1}{2}, \frac{1+\theta_2}{2}\right]$
$U2$	$\left[\frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_2^2}{2(2-(2-\alpha)\theta_2)}\right]$
$U3$	$\left[\frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_1^2}{2(2-(2-\alpha)\theta_1)}, \frac{2+\alpha-(2-\alpha)\theta_2^2}{2(2-(2-\alpha)\theta_2)}\right]$

3.3 Case 4

In this case the parametric family of auction models is determined by the values of the parameters belonging to the region $RC4 = \{\left(\gamma_3^2, \gamma_2^{\widehat{1}}, \gamma_1^{\widetilde{1}}, \gamma_1^2, \beta_1^{\widehat{2}}\right) \in [0, 1+\alpha] \times [0, \gamma_3^2] \times \left[0, \gamma_3^2 - \gamma_2^{\widehat{1}}\right] \times \left[\gamma_1^{\widetilde{1}}, 1-\alpha+\gamma_3^2\right] \times [0, 1-\alpha+\gamma_3^2-\gamma_1^2]\}$. There are twelve auction models on the vertices of this region. We will also consider the discriminatory auction model which is not located in any of the vertices of $RC4$.

Model	γ_3^2	$\gamma_2^{\widehat{1}}$	$\gamma_1^{\widetilde{1}}$	γ_1^2	$\beta_1^{\widehat{2}}$
$A1B1$	0	0	0	0	0
Vickrey	0	0	0	0	$1-\alpha$
$U1D1$	0	0	0	$1-\alpha$	0
UN	$1+\alpha$	0	0	0	0
$V1$	$1+\alpha$	0	0	0	2
$U2$	$1+\alpha$	0	0	2	0
$B2$	$1+\alpha$	0	$1+\alpha$	$1+\alpha$	0
$C2$	$1+\alpha$	0	$1+\alpha$	$1+\alpha$	$1-\alpha$
DI	$1+\alpha$	0	$1+\alpha$	2	0
$B3A2$	$1+\alpha$	$1+\alpha$	0	0	0
Uniform	$1+\alpha$	$1+\alpha$	0	0	2
$D2U3$	$1+\alpha$	$1+\alpha$	0	2	0
Discriminatory	$1+\alpha$	α	1	2	0

with the following bayesian Nash equilibria

Model/Equilibria	$\left[b_1^*(\theta_1), b_{\hat{1}}^*(\theta_1), b_2^*(\theta_2)\right]$
$A1B1$	$[\theta_1, \theta_1, \theta_2]$
Vickrey	$\left[b_{\hat{1}}^*(\theta_1), \theta_1, \theta_2\right]$ with $b_{\hat{1}}^*(\theta_1) \leq \theta_1 \forall \theta_1 \in [0,1]$ strictly monotone and differentiable
$U1D1$	$\left[b_{\hat{1}}^*(\theta_1), \frac{1+\theta_1}{2}, \frac{1+\theta_2}{2}\right]$ with $b_{\hat{1}}^*(\theta_1) \leq \theta_1 \forall \theta_1 \in [0,1]$ strictly monotone and differentiable
UN	$\left[\frac{-(1-\alpha)\theta_1+(1+\alpha)\theta_1^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}, \frac{-(1-\alpha)\theta_1+(1+\alpha)\theta_1^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}, \frac{-(1-\alpha)\theta_2+(1+\alpha)\theta_2^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}\right]$
$V1$	$\left[\frac{-(1-\alpha)\theta_1+(1+\alpha)\theta_1^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}, \frac{-(1-\alpha)\theta_1+(1+\alpha)\theta_1^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}, \frac{-(1-\alpha)\theta_2+(1+\alpha)\theta_2^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}\right]$
$U2$	$\left[\frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_1^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_1)}, \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_1^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_1)}, \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_2^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_2)}\right]$
$B2$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(1-\alpha)\theta_i+(1+\alpha)\left(1-e^{\frac{-(1-\theta_i)(1-\alpha)}{1+\alpha}}\right)}{1-\alpha}$
$C2$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(1-\alpha)\theta_i+(1+\alpha)\left(1-e^{\frac{-(1-\theta_i)(1-\alpha)}{1+\alpha}}\right)}{1-\alpha}$
DI	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_i^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_i)}$
$B3A2$	$[b^*(\theta_1), b^*(\theta_1), b^*(\theta_2)]$ where $b^*(\theta_i) = \frac{-(1-\alpha)\theta_i+(1+\alpha)\theta_i^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}$

Uniform	$\left[b_{\tilde{1}}^{*}(\theta_1), \frac{-(1-\alpha)\theta_1+(1+\alpha)\theta_1^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}, \frac{-(1-\alpha)\theta_2+(1+\alpha)\theta_2^{\frac{1-\alpha}{1+\alpha}}}{2\alpha}\right]$ with $b_{\tilde{1}}^{*}(\theta_1) \leq \frac{-(1-\alpha)\theta_1+(1+\alpha)\theta_1^{\frac{1-\alpha}{1+\alpha}}}{2\alpha} \forall\theta_1 \in [0,1]$ strictly monotone and differentiable
$D2U3$	$\left[b_{\tilde{1}}^{*}(\theta_1), \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_1^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_1)}, \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_2^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_2)}\right]$ with $b_{\tilde{1}}^{*}(\theta_1) \leq \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_1^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_1)} \forall\theta_1 \in [0,1]$ strictly monotone and differentiable
Discriminatory	$\left[\frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_1^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_1)}, \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_1^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_1)}, \frac{(1-\alpha)\left(3+\alpha-(1-\alpha)\theta_2^2\right)}{2(1-\alpha)(2-(1-\alpha)\theta_2)}\right]$

4 Expected revenue and payment

Let's see what these expressions are like for each demand size.

4.1 Case 2

Every bayesian Nash equilibrium in Case 2 has the following form

$$\left[b_{\tilde{1}}^{*}(\theta_1), b_{\hat{1}}^{*}(\theta_1), b_{\tilde{2}}^{*}(\theta_2)\right] = \left[b^{*}(\theta_1), b^{*}(\theta_1), b^{*}(\theta_2)\right]$$

so if an auction model on the vertices of $RC2$ is used and both companies bid with their strategies in equilibrium, the revenue for company i is reduced to

$$I_i(\theta_i,\theta_j) = \begin{cases} \gamma_1^2 b^{*}(\theta_i) + (1+\alpha-\gamma_1^2)b^{*}\left(\theta_j\right) & if \quad \theta_i < \theta_j \\ 0 & if \quad \theta_i > \theta_j \end{cases}$$

where $\gamma_1^2 \in [0, 1+\alpha]$, and the expected revenue for company i is:

$$\begin{aligned} P_i(\theta_i) &= E_{\theta_j}[I_i(\theta_i,\theta_j)] = \int_0^1 I_i(\theta_i,\theta_j)d\theta_j \\ &= (1-\theta_i)\,\gamma_1^2 b^{*}(\theta_i) + (1+\alpha-\gamma_1^2)\int_{\theta_i}^1 b^{*}\left(\theta_j\right)d\theta_j \\ &= \frac{(1+\alpha)(1-\theta_i^2)}{2} \end{aligned}$$

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in any auction model considered. Moreover, the payment the Market Operator expects to make is:

$$\begin{aligned} P_{MO} &= \sum_{i=1}^{2} E_{\theta_i}[P_i(\theta_i)] = 2\int_0^1 P_i(\theta_i)d\theta_i \\ &= \frac{2\,(1+\alpha)}{3} \end{aligned}$$

Remark 2. *Clearly the expected revenue for the companies and the payment the Market Operator expects to make are not dependent on the parameters, therefore we obtain an equivalence revenue result for every auction model belonging to the vertices of RC2.*

4.2 Case 3

Every bayesian Nash equilibrium in Case 3 has the following form

$$\left[b_1^*\left(\theta_1\right), b_1^*\left(\theta_1\right), b_2^*\left(\theta_2\right)\right] = \left[b^*\left(\theta_1\right), b^*\left(\theta_1\right), b^*\left(\theta_2\right)\right]$$

so if an auction model on the vertices of $RC3$ is used and both companies bid with their strategies in equilibrium, the revenue for company i is reduced to

$$I_i(\theta_i,\theta_j) = \begin{cases} \gamma_1^2 b^*\left(\theta_i\right) + \left(2-\alpha+\gamma_3^2-\gamma_1^2\right) b^*\left(\theta_j\right) + \alpha - \gamma_3^2 & if \quad \theta_i < \theta_j \\ \gamma_3^2 b^*\left(\theta_i\right) + \alpha - \gamma_3^2 & if \quad \theta_i > \theta_j \end{cases}$$

where $\gamma_1^2 \in \left[0, 2-\alpha+\gamma_3^2\right]$, $\gamma_3^2 \in [0,\alpha]$ and the expected revenue for company i is:

$$\begin{aligned} P_i(\theta_i) &= E_{\theta_j}[I_i(\theta_i,\theta_j)] = \int_0^1 I_i(\theta_i,\theta_j)d\theta_j \\ &= \left(\left(\gamma_3^2-\gamma_1^2\right)\theta_i + \gamma_1^2\right) b^*\left(\theta_i\right) + \alpha - \gamma_3^2 + \\ &+ \left(2-\alpha+\gamma_3^2-\gamma_1^2\right)\int_{\theta_i}^1 b^*\left(\theta_j\right) d\theta_j = \frac{(2-\alpha)(1-\theta_i^2)}{2} + \alpha \end{aligned}$$

in any auction model considered. Moreover, the payment the Market Operator expects to make is:

$$\begin{aligned} P_{MO} &= \sum_{i=1}^{2} E_{\theta_i}[P_i(\theta_i)] = 2\int_0^1 P_i(\theta_i)d\theta_i \\ &= \frac{4\,(1+\alpha)}{3} \end{aligned}$$

Remark 3. *Clearly the expected revenue for the companies and the payment the Market Operator expects to make are not dependent on the parameters, therefore we obtain an equivalence revenue result for every auction model belonging to the vertices of RC3.*

4.3 Case 4

Every bayesian Nash equilibrium in Case 4 has one of the following two forms

$$\left[b_{\tilde{1}}^{*}(\theta_1), b_{\hat{1}}^{*}(\theta_1), b_2^{*}(\theta_2)\right] = \left[b^{*}(\theta_1), b^{*}(\theta_1), b^{*}(\theta_2)\right]$$

or

$$\left[b_{\tilde{1}}^{*}(\theta_1), b_{\hat{1}}^{*}(\theta_1), b_2^{*}(\theta_2)\right] = \left[b_{\tilde{1}}^{*}(\theta_1), b^{*}(\theta_1), b^{*}(\theta_2)\right]$$

with $b_{\tilde{1}}^{*}(\theta_1) \leq b^{*}(\theta_1)\, \forall \theta_1 \in [0,1]$ strictly monotone and differentiable function, so if an auction model on the vertices of $RC4$ is used and both companies bid with their strategies in equilibrium, the revenue for company i is reduced in every case to

$$I_i(\theta_i, \theta_j) = \begin{cases} \gamma_1^2 b^{*}(\theta_i) + \left(1 - \alpha + \gamma_3^2 - \gamma_1^2\right) b^{*}\left(\theta_j\right) + 1 + \alpha - \gamma_3^2 & if \quad \theta_i < \theta_j \\ \gamma_3^2 b^{*}(\theta_i) + 1 + \alpha - \gamma_3^2 & if \quad \theta_i > \theta_j \end{cases}$$

where $\gamma_1^2 \in \left[0, 1 - \alpha + \gamma_3^2\right]$, $\gamma_3^2 \in [0, 1 + \alpha]$ and the expected revenue for company i is:

$$\begin{aligned} P_i(\theta_i) &= E_{\theta_j}[I_i(\theta_i, \theta_j)] = \int_0^1 I_i(\theta_i, \theta_j) d\theta_j \\ &= \left(\left(\gamma_3^2 - \gamma_1^2\right)\theta_i + \gamma_1^2\right) b^{*}(\theta_i) + 1 + \alpha - \gamma_3^2 + \\ &+ \left(1 - \alpha + \gamma_3^2 - \gamma_1^2\right) \int_{\theta_i}^1 b^{*}\left(\theta_j\right) d\theta_j = \frac{(1-\alpha)(1-\theta_i^2)}{2} + 1 + \alpha \end{aligned}$$

in any auction model considered. Moreover, the payment the Market Operator expects to make is:

$$\begin{aligned} P_{MO} &= \sum_{i=1}^{2} E_{\theta_i}[P_i(\theta_i)] = 2\int_0^1 P_i(\theta_i) d\theta_i \\ &= \frac{4\,(2+\alpha)}{3} \end{aligned}$$

Remark 4. *Clearly the expected revenue for the companies and the payment the Market Operator expects to make are not dependent on the parameters therefore, we obtain an equivalence revenue result for every auction model belonging to the vertices of $RC4$.*

Remark 5. *Alonso, E. and J. Tejada (2010) proved a revenue equivalence result when there are two suppliers with only one production unit.*

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Appendix

Proof of Proposition 1

The profit function of supplier 1 is:

$$B_1(\theta_1, m, M, b_2(\theta_2)) = \begin{cases} = \gamma_1^{\tilde{1}} m + \left(\gamma_1^2 - \gamma_1^{\tilde{1}}\right) M + \left(\phi_1 - \phi_3 + \gamma_3^2 - \gamma_1^2\right) b_2(\theta_2) + \phi_3 - \gamma_3^2 - \phi_1\theta_1 \\ \text{if} \quad b_2^{-1}(M) < \theta_2 \\ = \gamma_1^{\tilde{1}} m + \gamma_2^{\hat{1}} M + \left(\phi_2^1 - \phi_3 + \gamma_3^2 - \gamma_2^{\hat{1}} - \gamma_1^{\tilde{1}}\right) b_2(\theta_2) + \phi_3 - \gamma_3^2 - \phi_2^1\theta_1 \\ \text{if} \quad b_2^{-1}(m) < \theta_2 < b_2^{-1}(M) \\ = \left(\gamma_3^2 - \gamma_2^{\hat{1}}\right) m + \gamma_2^{\hat{1}} M + \phi_3 - \gamma_3^2 - \phi_3\theta_1 \quad \text{if} \quad \theta_2 < b_2^{-1}(m) \end{cases}$$

The profit function of supplier 2 is:

$$B_2(\theta_2, b_{\tilde{1}}(\theta_1), b_{\hat{1}}(\theta_1), b_2) = \begin{cases} = \gamma_1^2 b_2 + \left(\phi_1 - \phi_3 + \gamma_3^2 - \beta_1^{\hat{2}} - \gamma_1^2\right) b_{\tilde{1}}(\theta_1) + \beta_1^{\hat{2}} b_{\hat{1}}(\theta_1) + \phi_3 - \gamma_3^2 - \phi_1\theta_2 \\ \text{if} \quad \left(b_{\tilde{1}}\right)^{-1}(b_2) < \theta_1 \\ = \left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\hat{2}}\right) b_2 + \beta_1^{\hat{2}} b_{\hat{1}}(\theta_1) + \phi_3 - \gamma_3^2 - \phi_2^2\theta_2 \\ \text{if} \quad b_{\hat{1}}^{-1}(b_2) < \theta_1 < b_{\tilde{1}}^{-1}(b_2) \\ = \gamma_3^2 b_2 + \phi_3 - \gamma_3^2 - \phi_3\theta_2 \quad \text{if} \quad \theta_1 < b_{\hat{1}}^{-1}(b_2) \end{cases}$$

Supplier i knows its own type θ_i, but θ_j is a random variable, so the expected profit for supplier 1 is given by:

$$
\begin{aligned}
BM_1(\theta_1, b_{\widetilde{1}}, b_{\widehat{1}}, b_2^*(.)) &= \int_0^1 B_1(\theta_1, m, M, b_2^*(\theta_2))d\theta_2 = \\
&= \left(\gamma_1^{\widetilde{1}} m + \left(\gamma_1^2 - \gamma_1^{\widetilde{1}}\right) M + \phi_3 - \gamma_3^2 - \phi_1\theta_1\right)\left(1 - (b_2^*)^{-1}(M)\right) \\
&+ \left(\phi_1 - \phi_3 + \gamma_3^2 - \gamma_1^2\right) \int_{\left(b_2^*\right)^{-1}(M)}^1 b_2(\theta_2)\, d\theta_2 \\
&+ \left(\gamma_1^{\widetilde{1}} m + \gamma_2^{\widehat{1}} M + \phi_3 - \gamma_3^2 - \phi_2^1\theta_1\right)\left((b_2^*)^{-1}(M) - (b_2^*)^{-1}(m)\right) \\
&+ \left(\phi_2^1 - \phi_3 + \gamma_3^2 - \gamma_2^{\widehat{1}} - \gamma_1^{\widetilde{1}}\right) \int_{\left(b_2^*\right)^{-1}(m)}^{\left(b_2^*\right)^{-1}(M)} b_2(\theta_2)\, d\theta_2 \\
&+ \left(\left(\gamma_3^2 - \gamma_2^{\widehat{1}}\right) m + \gamma_2^{\widehat{1}} M + \phi_3 - \gamma_3^2 - \phi_3\theta_1\right)(b_2^*)^{-1}(m) = \\
&= \gamma_1^{\widetilde{1}} m + \left(\gamma_1^2 - \gamma_1^{\widetilde{1}}\right) M + \phi_3 - \gamma_3^2 - \phi_1\theta_1 \\
&+ (b_2^*)^{-1}(M)\left(\left(\gamma_2^{\widehat{1}} - \gamma_1^2 + \gamma_1^{\widetilde{1}}\right) M + \left(\phi_1 - \phi_2^1\right)\theta_1\right) \\
&+ (b_2^*)^{-1}(m)\left(\left(-\gamma_1^{\widetilde{1}} + \gamma_3^2 - \gamma_2^{\widehat{1}}\right) m + \left(\phi_2^1 - \phi_3\right)\theta_1\right) \\
&+ \left(\phi_1 - \phi_3 + \gamma_3^2 - \gamma_1^2\right) \int_{\left(b_2^*\right)^{-1}(M)}^1 b_2(\theta_2)\, d\theta_2 \\
&+ \left(\phi_2^1 - \phi_3 + \gamma_3^2 - \gamma_2^{\widehat{1}} - \gamma_1^{\widetilde{1}}\right) \int_{\left(b_2^*\right)^{-1}(m)}^{\left(b_2^*\right)^{-1}(M)} b_2(\theta_2)\, d\theta_2
\end{aligned}
$$

Where $m = \min(b_{\widetilde{1}}, b_{\widehat{1}})$ and $M = max(b_{\widetilde{1}}, b_{\widehat{1}})$. Thus $\left(b_{\widetilde{1}}, b_{\widehat{1}}\right)$ is the best bid for company 1 if it maximizes the expected profit, given that its type is θ_1. First we are going to suppose that supplier 1 makes different bids with its production units. Derivation with respect to $b_{\widetilde{1}}$ and $b_{\widehat{1}}$ (one of them is m and the other is M) yields

$$
\begin{aligned}
\frac{\partial}{\partial m} BM_1(\theta_1, m, M, b_2^*(.)) &= \\
&= \gamma_1^{\widetilde{1}} + \left(-\gamma_1^{\widetilde{1}} + \gamma_3^2 - \gamma_2^{\widehat{1}}\right)(b_2^*)^{-1}(m) \\
&+ \left(\phi_3 - \phi_2^1\right)(m - \theta_1)\frac{d}{dm}(b_2^*)^{-1}(m)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial M} BM_1(\theta_1, m, M, b_2^*(.)) &= \\
&= \left(\gamma_1^2 - \gamma_1^{\widetilde{1}}\right) + \left(\gamma_2^{\widehat{1}} - \gamma_1^2 + \gamma_1^{\widetilde{1}}\right)(b_2^*)^{-1}(M) \\
&+ \left(\phi_2^1 - \phi_1\right)(M - \theta_1)\frac{d}{dM}(b_2^*)^{-1}(M)
\end{aligned}
$$

If bidder 1 makes the lowest bid with the production unit $\widetilde{1}$ then as $b_{\widetilde{1}}(\theta_1) = m$ and $b_{\widehat{1}}(\theta_1) = M \Leftrightarrow \theta_1 = b_{\widetilde{1}}^{-1}(m) = b_{\widetilde{1}}^{-1}(M)$, replacing and setting the above equations to zero, we obtain the following differential equations:

$$\begin{cases} \gamma_1^{\widetilde{1}} + \left(-\gamma_1^{\widetilde{1}} + \gamma_3^2 - \gamma_2^{\widehat{1}}\right)\left(b_2^*\right)^{-1}(m) + \\ + \left(\phi_3 - \phi_2^1\right)\left(m - \left(b_{\widetilde{1}}^*\right)^{-1}(m)\right)\frac{d}{dm}\left(b_2^*\right)^{-1}(m) = 0 \\ \left(\gamma_1^2 - \gamma_1^{\widetilde{1}}\right) + \left(\gamma_2^{\widehat{1}} - \gamma_1^2 + \gamma_1^{\widetilde{1}}\right)\left(b_2^*\right)^{-1}(M) + \\ + \left(\phi_2^1 - \phi_1\right)\left(M - \left(b_{\widehat{1}}^*\right)^{-1}(M)\right)\frac{d}{dM}\left(b_2^*\right)^{-1}(M) = 0 \end{cases}$$

evaluating both on t we obtain

$$\begin{cases} \gamma_1^{\widetilde{1}} + \left(-\gamma_1^{\widetilde{1}} + \gamma_3^2 - \gamma_2^{\widehat{1}}\right)\left(b_2^*\right)^{-1}(t) + \\ + \left(\phi_3 - \phi_2^1\right)\left(t - \left(b_{\widetilde{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}\left(b_2^*\right)^{-1}(t) = 0 \\ \left(\gamma_1^2 - \gamma_1^{\widetilde{1}}\right) + \left(\gamma_2^{\widehat{1}} - \gamma_1^2 + \gamma_1^{\widetilde{1}}\right)\left(b_2^*\right)^{-1}(t) + \\ + \left(\phi_2^1 - \phi_1\right)\left(t - \left(b_{\widehat{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}\left(b_2^*\right)^{-1}(t) = 0 \end{cases}$$

If bidder 1 makes the same bid with both production units then we obtain only one differential equation and it is:

$$\gamma_1^2 + \left(\gamma_3^2 - \gamma_1^2\right)\left(b_2^*\right)^{-1}(t) + \left(\phi_3 - \phi_1\right)\left(t - \left(b_1^*\right)^{-1}(t)\right)\frac{d}{dt}\left(b_2^*\right)^{-1}(t) = 0$$

On the other hand, the expected profit for supplier 2 is given by

$$BM_2(\theta_2, b_{\widetilde{1}}^*(.), b_{\widehat{1}}^*(.), b_2) = \int_0^1 B_2(\theta_2, b_{\widetilde{1}}^*(\theta_1), b_{\widehat{1}}^*(\theta_1), b_2)d\theta_2 =$$

$$\begin{aligned} &= \left(\gamma_1^2 b_2 + \phi_3 - \gamma_3^2 - \phi_1\theta_2\right)\left(1 - (b_{\widetilde{1}}^*)^{-1}(b_2)\right) \\ &+ \left(\phi_1 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)\int_{(b_{\widetilde{1}}^*)^{-1}(b_2)}^1 b_{\widetilde{1}}(\theta_1)\,d\theta_1 + \beta_1^{\widehat{2}}\int_{(b_{\widehat{1}}^*)^{-1}(b_2)}^1 b_{\widehat{1}}(\theta_1)\,d\theta_1 \\ &+ \left(\left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}}\right)b_2 + \phi_3 - \gamma_3^2 - \phi_2^2\theta_2\right)\left((b_{\widetilde{1}}^*)^{-1}(b_2) - (b_{\widehat{1}}^*)^{-1}(b_2)\right) \\ &+ \left(\gamma_3^2 b_2 + \phi_3 - \gamma_3^2 - \phi_3\theta_2\right)(b_{\widehat{1}}^*)^{-1}(b_2) \\ &= \gamma_1^2 b_2 + \phi_3 - \gamma_3^2 - \phi_1\theta_2 \\ &+ (b_{\widetilde{1}}^*)^{-1}(b_2)\left(\left(\phi_1 - \phi_2^2\right)\theta_2 + \left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)b_2\right) \\ &+ (b_{\widehat{1}}^*)^{-1}(b_2)\left(-\left(\phi_2^2 - \phi_3 - \beta_1^{\widehat{2}}\right)b_2 + \left(\phi_2^2 - \phi_3\right)\theta_2\right) \\ &+ \left(\phi_1 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)\int_{(b_{\widetilde{1}}^*)^{-1}(b_2)}^1 b_{\widetilde{1}}(\theta_1)\,d\theta_1 + \beta_1^{\widehat{2}}\int_{(b_{\widehat{1}}^*)^{-1}(b_2)}^1 b_{\widehat{1}}(\theta_1)\,d\theta_1 \end{aligned}$$

Derivation with respect to b_2 yields

$$\begin{aligned} &\frac{\partial}{\partial b_2}BM_2(\theta_2, b_{\widetilde{1}}^*(.), b_{\widehat{1}}^*(.), b_2) = \\ = \quad &\gamma_1^2 + \left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)(b_{\widetilde{1}}^*)^{-1}(b_2) - \left(\phi_2^2 - \phi_3 - \beta_1^{\widehat{2}}\right)(b_{\widehat{1}}^*)^{-1}(b_2) \\ &+ \left(\phi_2^2 - \phi_1\right)(b_2 - \theta_2)\left((b_{\widetilde{1}}^*)^{-1}(b_2)\right)' + \left(\phi_3 - \phi_2^2\right)(b_2 - \theta_2)\left((b_{\widehat{1}}^*)^{-1}(b_2)\right)' \end{aligned}$$

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As $b_2^{-1}(b_2) = \theta_2 \Longleftrightarrow b_2(\theta_2) = b_2$, replacing and setting the above equations to zero, we obtain the following differential equation:

$$\begin{aligned}
0 = {} & \gamma_1^2 + \left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)(b_{\tilde{1}}^*)^{-1}(b_2^*(\theta_2)) \\
& - \left(\phi_2^2 - \phi_3 - \beta_1^{\widehat{2}}\right)(b_{\widehat{1}}^*)^{-1}(b_2^*(\theta_2)) \\
& + \left(\phi_2^2 - \phi_1\right)(b_2^*(\theta_2) - \theta_2)\left((b_{\tilde{1}}^*)^{-1}\right)'(b_2^*(\theta_2)) \\
& + \left(\phi_3 - \phi_2^2\right)(b_2^*(\theta_2) - \theta_2)\left((b_{\widehat{1}}^*)^{-1}\right)'(b_2^*(\theta_2))
\end{aligned}$$

Evaluating on t

$$\begin{aligned}
0 = {} & \gamma_1^2 + \left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)(b_{\tilde{1}}^*)^{-1}(t) - \left(\phi_2^2 - \phi_3 - \beta_1^{\widehat{2}}\right)(b_{\widehat{1}}^*)^{-1}(t) \\
& + \left(\phi_2^2 - \phi_1\right)\left(t - (b_2^*)^{-1}(t)\right)\frac{d}{dt}(b_{\tilde{1}}^*)^{-1}(t) \\
& + \left(\phi_3 - \phi_2^2\right)\left(t - (b_2^*)^{-1}(t)\right)\frac{d}{dt}(b_{\widehat{1}}^*)^{-1}(t)
\end{aligned}$$

Then we have obtained two differential systems, if bidder 1 makes two bids then the system is

$$(S1)\left\{\begin{array}{l}
\gamma_1^{\tilde{1}} + \left(-\gamma_1^{\tilde{1}} + \gamma_3^2 - \gamma_2^{\widehat{1}}\right)(b_2^*)^{-1}(t) \\
+ (\phi_3 - \phi_2^1)\left(t - \left(b_{\tilde{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}(b_2^*)^{-1}(t) = 0 \\
\left(\gamma_1^2 - \gamma_1^{\tilde{1}}\right) + \left(\gamma_2^{\widehat{1}} - \gamma_1^2 + \gamma_1^{\tilde{1}}\right)(b_2^*)^{-1}(t) \\
+ (\phi_2^1 - \phi_1)\left(t - \left(b_{\widehat{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}(b_2^*)^{-1}(t) = 0 \\
\gamma_1^2 + \left(\phi_2^2 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2\right)(b_{\tilde{1}}^*)^{-1}(t) - \left(\phi_2^2 - \phi_3 - \beta_1^{\widehat{2}}\right)(b_{\widehat{1}}^*)^{-1}(t) \\
+ (\phi_2^2 - \phi_1)\left(t - (b_2^*)^{-1}(t)\right)\frac{d}{dt}(b_{\tilde{1}}^*)^{-1}(t) \\
+ (\phi_3 - \phi_2^2)\left(t - (b_2^*)^{-1}(t)\right)\frac{d}{dt}(b_{\widehat{1}}^*)^{-1}(t) = 0
\end{array}\right.$$

And if bidder 2 makes the same bid with both production units then the system is:

$$(S2)\left\{\begin{array}{l}
\gamma_1^2 + (\gamma_3^2 - \gamma_1^2)(b_2^*)^{-1}(t) + (\phi_3 - \phi_1)\left(t - (b_1^*)^{-1}(t)\right)\frac{d}{dt}(b_2^*)^{-1}(t) = 0 \\
\gamma_1^2 + (\gamma_3^2 - \gamma_1^2)(b_1^*)^{-1}(t) + (\phi_3 - \phi_1)\left(t - (b_2^*)^{-1}(t)\right)\frac{d}{dt}(b_1^*)^{-1}(t) = 0
\end{array}\right.$$

We have the following regions:

Region I If $\phi_3 = \phi_2^1$, $\phi_1 = \phi_2^2$, $\gamma_1^{\tilde{1}} = 0$, $\gamma_2^{\widehat{1}} = \gamma_3^2$, $\phi_1 - \phi_3 + \gamma_3^2 - \beta_1^{\widehat{2}} - \gamma_1^2 = 0$ then the system $(S1)$ is reduced to

$$\left\{\begin{array}{l}
\gamma_1^2 + (\gamma_3^2 - \gamma_1^2)(b_2^*)^{-1}(t) + (\phi_3 - \phi_1)\left(t - \left(b_{\widehat{1}}^*\right)^{-1}(t)\right)\frac{d}{dt}(b_2^*)^{-1}(t) = 0 \\
\gamma_1^2 + (\gamma_3^2 - \gamma_1^2)(b_{\widehat{1}}^*)^{-1}(t) + (\phi_3 - \phi_1)\left(t - (b_2^*)^{-1}(t)\right)\frac{d}{dt}(b_{\widehat{1}}^*)^{-1}(t) = 0
\end{array}\right.$$

and with the conditions $b^*_2(1) = b^*_{\widehat{1}}(1) = 1$, it follows that there exist infinite bayesian Nash equilibrium given by

$$\left(b^*_{\widetilde{1}}(\theta_1), b^*_{\widehat{1}}(\theta_1), b^*_2(\theta_2)\right) = \left(b^*_{\widetilde{1}}(\theta_1), b^*(\theta_1), b^*(\theta_2)\right)$$

where $b^*_{\widetilde{1}}(\theta_1)$ is any strictly monotone and differentiable function verifying $b^*_{\widetilde{1}}(\theta_1) \leq b^*(\theta_1)\ \forall\theta_1 \in [0,1]$.

Region II Otherwise and if one of the following expressions is true

•$\gamma_2^{\widehat{1}} \neq 0$ or $\gamma_3^2 \neq 0$

•$\gamma_2^{\widehat{1}} = 0$ and $\gamma_1^{\widetilde{1}} = \gamma_1^2$

•$\gamma_2^{\widehat{1}} = \gamma_3^2 = 0$ and $\gamma_1^2 = (1+k)\,\gamma_1^{\widetilde{1}}$, $(1+k)\,\phi_2^1 = \phi_1 + k\phi_3$, for any $k \in \Re$

Then 1 makes the same bid with the two production units. So the system is reduced to $(S2)$ and with the conditions $b^*_2(1) = b^*_{\widehat{1}}(1) = 1$, it follows that there exist unique symmetric bayesian Nash equilibrium given by

$$\left(b^*_{\widetilde{1}}(\theta_1), b^*_{\widehat{1}}(\theta_1), b^*_2(\theta_2)\right) = (b^*(\theta_1), b^*(\theta_1), b^*(\theta_2))$$

Both in **Region I** as in **Region II,** $b^*(\theta_i)$ is a particular solution of the differential equation

$$\left(\gamma_1^2 + \left(\gamma_3^2 - \gamma_1^2\right)\theta_i\right)(b^*)'(\theta_i) - (\phi_1 - \phi_3)\, b^*(\theta_i) = -(\phi_1 - \phi_3)\,\theta_i$$

with $b^*(1) = 1$. Solving the differential equation we obtain

a) If $\gamma_1^2 = \gamma_3^2 = 0$, then

$$b^*(\theta_i) = \theta_i$$

b) If $\gamma_3^2 = 0$ and $\gamma_1^2 \neq 0$, then

$$b^*(\theta_i) = \frac{(\phi_1 - \phi_3)\,\theta_i + \gamma_1^2}{\gamma_1^2 + \phi_1 - \phi_3}$$

c) If $\gamma_1^2 \neq \gamma_3^2 \neq 0$ then

$$b^*(\theta_i) = \frac{(\phi_3 - \phi_1)\,\theta_i - \gamma_1^2 + \left(\gamma_3^2\right)^{\frac{\gamma_3^2 - \gamma_1^2 + \phi_3 - \phi_1}{\gamma_3^2 - \gamma_1^2}} \left(\gamma_1^2 + \left(\gamma_3^2 - \gamma_1^2\right)\theta_i\right)^{\frac{\phi_1 - \phi_3}{\gamma_3^2 - \gamma_1^2}}}{\gamma_3^2 - \gamma_1^2 + \phi_3 - \phi_1}$$

d) If $\gamma_1^2 = \gamma_3^2 \neq 0$ then

$$b^*(\theta_i) = \theta_i + \frac{\gamma_1^2\left(1 - e^{\frac{-(1-\theta_i)(\phi_1 - \phi_3)}{\gamma_1^2}}\right)}{\phi_1 - \phi_3}$$

Region III If the parameters don't belong to **Region I** or **Region II** then we can only say that the bayesian Nash equilibria are solutions of the following system of differential equations:

$$\begin{cases} \gamma_{1}^{\widetilde{1}}\left(1-\left(b_{2}^{*}\right)^{-1}(t)\right)+\left(\phi_{3}-\phi_{2}^{1}\right)\left(t-\left(b_{\widetilde{1}}^{*}\right)^{-1}(t)\right)\frac{d}{dt}\left(b_{2}^{*}\right)^{-1}(t)=0 \\ \left(\gamma_{1}^{2}-\gamma_{1}^{\widetilde{1}}\right)\left(1-\left(b_{2}^{*}\right)^{-1}(t)\right)+\left(\phi_{2}^{1}-\phi_{1}\right)\left(t-\left(b_{\widehat{1}}^{*}\right)^{-1}(t)\right)\frac{d}{dt}\left(b_{2}^{*}\right)^{-1}(t)=0 \\ \gamma_{1}^{2}\left(1-(b_{\widetilde{1}}^{*})^{-1}(t)\right)+\left(\phi_{2}^{2}-\phi_{3}-\beta_{1}^{\widehat{2}}\right)\left((b_{\widetilde{1}}^{*})^{-1}(t)-(b_{\widehat{1}}^{*})^{-1}(t)\right) \\ +\left(\phi_{2}^{2}-\phi_{1}\right)\left(t-\left(b_{2}^{*}\right)^{-1}(t)\right)\frac{d}{dt}(b_{\widetilde{1}}^{*})^{-1}(t) \\ +\left(\phi_{3}-\phi_{2}^{2}\right)\left(t-\left(b_{2}^{*}\right)^{-1}(t)\right)\frac{d}{dt}(b_{\widehat{1}}^{*})^{-1}(t)=0 \end{cases}$$

■

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