Investigación

Explicit formulas for the Euler's phi function and the counting of primes

Fórmulas explícitas para la función phi de Euler y el conteo de primos

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Revista de Investigación

Volumen XV, pp. 013-022, ISSN 2174-0410
Recepción: 02 Mar'24; Aceptación: 22 Oct'24

1 de abril de 2025

Abstract

An explicit formula which characterizes the pairs of integers that are relatively prime is obtained. It doesn't require the knowledge of the prime factors of the arguments, what allows to construct other explicit formulas for the Euler's totient function and the Prime counting function.

Keywords: Prime counting function, Euler's totient function, Prime number, Relatively prime numbers, Explicit formula.

Resumen

Se obtiene aquí una fórmula explícita que caracteriza los pares de enteros que son primos relativos. No requiere del conocimiento de los factores primos de los argumentos, lo que permite construir otras fórmulas explícitas para la función phi de Euler y la contadora de primos.

Palabras Clave: Función contadora de primos, Función phi de Euler, Número primo, Primos relativos, Fórmula explícita.

1. Introduction

It is deduced in the first place a characteristic function for pairs of relatively prime numbers (that is, without common prime factors) which will be useful also to obtain later exact formulas for the known as Euler's phi function and Prime counting function.

For the first one, also called Totient function, that gives the amount of positive integers which are -relatively- prime with some given integer greater than them it is well known the Euler's formula

$$\varphi(n) \;=\; n \prod_{p|n}\left(1-\frac{1}{p}\right)$$

meaning that (the primes) p must be divisors of n, but obtaining these divisors is a different task; it is possible also to write φ as the real part of the discrete Fourier transform of the *GCD* -greatest common divisor- evaluated at 1 (as reminded for instance in [6] by Schramm) but that formula requires all the *GCD* of the argument with the previous numbers:

$$\varphi(n) \;=\; \sum_{k=1}^{n} GCD(k,n)\,\cos\left(\frac{2\,\pi\,k}{n}\right)$$

Regarding the asymptotic behaviour of the function there are many articles but it is hard to find exact expressions like these. Here two formulas are deduced with the additional advantage that they don't need the prime factors of the argument nor auxiliary algorithms, so that they allow a direct evaluation by substitution (as long as the roundings to certain decimal order don't affect the correct values of the integer parts involved).

In relation to the Prime counting function, $\pi(x)$, which gives the amount of primes not greater than a given number, asymptotic formulas have also been found and some other exact ones with mainly a theoretical interest, given that they don't provide very efficient methods for big numbers due to speed or precision. Involving only elementary functions for a direct calculation, as the ones shown here, there is one quite known by Willans [9], based directly on the characterization of primes by the 'Wilson's theorem' and its inverse, although it includes factorials in the arguments, or gamma function in Connes' version [1], both with the inconvenient of their fast growth. The formulas obtained here provide a different approach with arguments that maybe can help also to clarify relations among primes, modular arithmetic and trigonometric functions.

2. Characterizing relative primes

Definition. For every integer $i > 1$, the real (i-*th*) function *rp-wave* (of a real variable x) is defined in the next way:

$$v_i(x) \;\equiv \prod_{k=1}^{\lfloor i/2 \rfloor}\left|\sin^{-1}\left(\frac{k\,\pi}{i}\right)\sin\left(\frac{k\,\pi\,x}{i}\right)\right| \;=\; \alpha_i \prod_{k=1}^{\lfloor i/2 \rfloor}\left|\sin\left(\frac{k\,\pi\,x}{i}\right)\right| \tag{2.1}$$

where $\alpha_i \equiv \left(\prod_{k=1}^{\lfloor i/2 \rfloor}\sin\left(\frac{k\,\pi}{i}\right)\right)^{-1}$, $\lfloor\ \rfloor$ means floor function and the bars, absolute value.

Each factor $\left|\sin\ (k\ \pi\ x\ /\ i)\right|$ will be referred to as the *k component* of the rp-wave.

It will be enough to focus on positive integer -natural- values of x. Next, we prove that, at every integer $x = j > 0$, the rp-waves can take only the value 0 or 1 (this, when they are relatively prime numbers). So we have a way to characterize the pairs of integers without prime factors in common: it will suffice to take also i as a variable in (2.1), allowing for x only natural values.

Proposition 1. For $1 < i \in \mathbb{Z}$ and $0 < j \in \mathbb{Z}$:

$$v_i(j) = 0 \quad \Leftrightarrow \quad GCD\ (i\ ,\ j)\ >\ 1 \tag{2.2}$$

Proof. Remind that the *LCM* -least common multiple- equals the product of two positive integers if and only if their *GCD* is 1.

From left to right, if the product of sines vanishes in (2.1), it means that at least one of the fractions $k \cdot j / i$ in the arguments has an integer value, which implies for some k that i is divisor of $k \cdot j$. But then $GCD(i, j) = 1$ would lead to a contradiction because i should in fact be a divisor of k, which is not greater than $i / 2$ according with (2.1). Conversely, when $GCD(i, j)$ is greater than 1, then $LCM(i, j)$ is less than $j \cdot i$, what means that the product of j by some integer k not greater than $i / 2$ is a multiple of i and the fraction $k \cdot j / i$ with such value of k in (2.1) is an integer and the corresponding sine vanishes. □

Proposition 2. For $1 < i \in \mathbb{Z}$ and $0 < j \in \mathbb{Z}$:

$$v_i(j) = 1 \quad \Leftrightarrow \quad GCD\ (i\ ,\ j)\ =\ 1 \tag{2.3}$$

Proof. Using proposition 1 it is enough to prove the implication from right to left, but this will require several steps; so we make first some observations relating these functions.

Leaving apart the constant α_i (a normalizing one, as we see later), the *i-th* rp-wave is a product of components which are absolute values of sine functions and, due to their arguments, have periods (for x) that are all fractions of i. So their product is another periodic non negative function with period i, which is precisely the period of the component $k = 1$, and it vanishes -at least- at the extremes of the interval $[0, i]$. It is also true that each component is a symmetric function on that interval with respect to its central abscissa $i/2$ (independently of the parities of i and k) and in the same way it is also symmetric the rp-wave, with same images at same distances of that center.

Besides, the rp-wave is injective at most on intervals of amplitude $i / 2$ as $[0, i / 2]$, due to the combination of symmetry and periodicity. So, it can take at most $\lfloor i\ /\ 2 \rfloor + 1$ different values for integer abscissas in general because on $[0, i / 2]$ there are only $\lfloor i\ /\ 2 \rfloor + 1$ different integer values for x. Therefore, as α_i is the inverse of the product of the components at $x = 1$, it is enough to check that the *i-th* rp-wave has the same value at any other positive integer that, as 1, is relatively prime with i (what means that it is normalized at natural abscissas because at the ones not prime with i it must vanish, according to the first proposition).

For $x = 1$ the arguments of the sines of the components are the successive multiples of the quotient π / i, until $\lfloor i / 2 \rfloor \cdot \pi / i$, so that the sines take positive increasing values, all different. And that π / i is the minimum possible increment for the arguments in general when x is a positive integer, so that at each other natural x every component must vanish or equal one of those same $\lfloor i / 2 \rfloor$ values of the sine function, thanks to the symmetry and periodicity of its absolute value. As a result, in order to check the coincidence of values at all x that are prime with the index i of the considered rp-wave, it will suffice to show that all the components take different values at each of those integers x ($\equiv j$). This is enough because -according to the first proposition- none of the components can vanish there and so each one of them must take one of those distinct $\lfloor i / 2 \rfloor$ positive values, which means that if the components are all different, necessarily each of those values will be a factor in the total product (2.1) of the rp-wave, because it has precisely $\lfloor i / 2 \rfloor$ components, so that it must have the same value as for $x = 1$.

It is worth noting also that none of the components can equal the value 1 for integer x if i is odd because all k are then less than $i / 2$. But 1 is the value of the last component, the one with $k = i / 2$, if the index i of the rp-wave is even for an integer $x = j$ that is prime with i (and necessarily odd). This is obvious not only for $x = 1$, and no other component can equal 1 at these abscissas because $i / 2$ in these cases is also prime with j, so that there is no other integer less than $i / 2$ which gives a multiple of $i / 2$ when multiplied by that j (what we need to have a value 1 or -1 of the sine). We will remind this later and focus for the moment on the other components, those with $k < i / 2$ independently of the parity of i (components with sine's arguments not multiples of $\pi / 2$ for any integer x prime with the index i, what we assume until the last paragraph).

Now, in relation with the fact that the minimum possible increment for the arguments of the sines for integer x is the quotient π / i, we will use the known as 'cancellation property' in modular arithmetic: for $i > 1$, prime with j, $a \cdot j = b \cdot j$ $(Mod\ (i))$ implies $a = b$ $(Mod\ (i))$, where all numbers are supposed to be integers. Next we distinguish two cases for each natural x: components k with sine's arguments which belong to odd angular quadrants (these are the arguments between 0 and $\pi / 2$ except for maybe a difference of a multiple of π, that is, with even value of $\lfloor 2 k x / i \rfloor$) and components with sine's arguments in even angular quadrants (so, with odd value of $\lfloor 2 k x / i \rfloor$), being the first quadrant $(0, \pi / 2)$, as usual.

It is clear that two different components with sine's arguments $(k' \pi j / i)$ and $(k'' \pi j / i)$ both in odd quadrants take different values, not only when $j = 1$, because their respective k are distinct numbers which must give distinct results $Mod(i)$ when multiplied by that j, according to the cited property under the assumptions, and this implies different arguments also after any difference of a multiple of π in one of them. As the absolute value of the sine is strictly increasing with the argument in those quadrants, also the values of the components must differ. In a similar way we can deduce that two distinct components with arguments both in even quadrants have different values (now the absolute value of the sine decreases strictly with the argument).

For the moment we have used the fact that the first positive integers until $\lfloor i / 2 \rfloor$ are part of a 'complete residue system' modulo i, but the absolute value of the sine is injective at most in intervals of amplitude $\pi / 2$. In the case of two components with arguments $(k' \pi j / i)$ and

($k'' \pi j / i$) that belong to quadrants with different parities, we will make use of the identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, reasoning by reductio ad absurdum. If we suppose that two such components are equal for that $x = j$, relative prime with i, then there would be two possibilities for the pair of arguments, because the corresponding sines -before absolute values- would be either equal or opposite numbers. But both of them lead then to a contradiction; being for instance $k' < k''$ ($< i / 2$ as we still exclude the value 1 for the components) without loss of generality, the two possibilities are:

a) If $\sin(k' \pi j / i) \equiv \sin A = \sin(k'' \pi j / i) \equiv \sin B$ then, due to the opposite parities of the quadrants, it is also $\cos B = -\cos A$ and $\sin(A+B) = \sin A \cdot (-\cos A + \cos A) = 0$, so that the argument $A + B = (k' j + k'' j) \cdot \pi / i$ must be an integer multiple of π.

b) If $\sin(k' \pi j / i) \equiv \sin A = -\sin(k'' \pi j / i) \equiv -\sin B$, then it must be $\cos B = \cos A$ and $\sin(A+B) = \sin A \cdot (\cos A - \cos A) = 0$, reaching the same conclusion.

Both cases are impossible because if $(k'j + k''j) / i$ is integer then it is $(k' + k'') \cdot j = q \cdot i$, with q being also an integer, but j is supposed to be relative prime with i, so that all the prime factors of i (including possible repetitions) should be included in the factor ($k' + k''$), which is false because it is smaller than i due to the assumption $k' < k'' < i / 2$.

Now we only have to remind, as stated previously, that for an even i the component $k = i / 2$ is the only one which can equal 1 for integer x, only when it is odd. So, including also this component it is clear that at any x that is prime with i, where the $\lfloor i / 2 \rfloor$ components can't vanish, their values must be all different and so they form necessarily the same set as for $x = 1$. Hence, according to the previous reasoning, the rp-wave also equals 1 there. □

3. Formulas for the Euler's φ function

Lemma. Let i, j be integers greater than 1. Then: $v_i(j) = v_j(i)$

Proof. Straightforward from propositions 1 and 2 and the symmetry of the *GCD* with respect to its arguments. □

Using (2.1) and this symmetry we obtain two distinct expressions for the Totient function, which counts the positive integers that are relatively prime with another greater one.

Theorem 1. For $2 < n \in \mathbb{Z}$:

$$\varphi(n) = 1 + \sum_{i=2}^{n-1} \prod_{k=1}^{\lfloor i / 2 \rfloor} \left| \sin^{-1}\left(\frac{k \pi}{i}\right) \sin\left(\frac{n k \pi}{i}\right) \right| \tag{3.1}$$

Proof. Straightforward from propositions 1 and 2 and the definition of the function φ (with the sum of the successive rp-waves evaluated at that n, adding 1 because they are defined from index 2). □

Examples. We can check for instance these first values:

$$\varphi(3) \;=\; 1+\left|\sin^{-1}\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)\right| = 1+1=2$$

$$\varphi(4) \;=\; 1+\left|\sin^{-1}\left(\frac{\pi}{2}\right)\sin\left(\frac{4\pi}{2}\right)\right|+\left|\sin^{-1}\left(\frac{\pi}{3}\right)\sin\left(\frac{4\pi}{3}\right)\right| = 1+0+1=2$$

$$\varphi(5) \;=\; 1+\left|\sin^{-1}\left(\frac{\pi}{2}\right)\sin\left(\frac{5\pi}{2}\right)\right|+\left|\sin^{-1}\left(\frac{\pi}{3}\right)\sin\left(\frac{5\pi}{3}\right)\right|+\left|\sin^{-1}\left(\frac{\pi}{4}\right)\sin\left(\frac{5\pi}{4}\right)\sin^{-1}\left(\frac{2\pi}{4}\right)\sin\left(\frac{10\pi}{4}\right)\right|$$
$$=1+1+1+1=4$$

$$\varphi(6) \;=\; 1+\left|\sin^{-1}\left(\frac{\pi}{2}\right)\sin\left(\frac{6\pi}{2}\right)\right|+\left|\sin^{-1}\left(\frac{\pi}{3}\right)\sin\left(\frac{6\pi}{3}\right)\right|+\left|\sin^{-1}\left(\frac{\pi}{4}\right)\sin\left(\frac{6\pi}{4}\right)\sin^{-1}\left(\frac{2\pi}{4}\right)\sin\left(\frac{12\pi}{4}\right)\right|$$
$$+\left|\sin^{-1}\left(\frac{\pi}{5}\right)\sin\left(\frac{6\pi}{5}\right)\sin^{-1}\left(\frac{2\pi}{5}\right)\sin\left(\frac{12\pi}{5}\right)\right| = 1+0+0+0+1=2$$

Theorem 2. For $2 < n \in \mathbb{Z}$:

$$\varphi(n) \;=\; 1 \;+\; \sum_{j=2}^{n-1}\prod_{k=1}^{\lfloor n\,/\,2\rfloor}\left|\sin^{-1}\left(\frac{k\;\pi}{n}\right)\sin\left(\frac{j\;k\;\pi}{n}\right)\right| \tag{3.2}$$

Proof. Straightforward from propositions 1 and 2 and the definition of the function φ (now with the sum of the n-th rp-wave evaluated at the successive natural numbers, from 2, adding 1, which is the value of any rp-wave at 1). □

Examples. Now we will obtain in this way the first two previous values:

$$\varphi(3) \;=\; 1+\left|\sin^{-1}\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\right| = 1+1=2$$

$$\varphi(4) \;=\; 1+\left|\sin^{-1}\left(\frac{\pi}{4}\right)\sin\left(\frac{2\pi}{4}\right)\sin^{-1}\left(\frac{2\pi}{4}\right)\sin\left(\frac{4\pi}{4}\right)\right|+\left|\sin^{-1}\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right)\sin^{-1}\left(\frac{2\pi}{4}\right)\sin\left(\frac{6\pi}{4}\right)\right|$$
$$=1+0+1=2$$

It is clear that for an even n we could skip the addends with smaller even indexes, but this would make the expressions less general. As we announced, in principle they have mainly a conceptual interest. A bit more about this is said after the next section.

4. Identifying and counting primes

Proposition 3. Let $j > 3$ be an integer:

$$\prod_{i=2}^{\lfloor\sqrt{j}\rfloor} v_i(j) \quad\text{and}\quad \prod_{i=2}^{\lfloor\sqrt{j}\rfloor} v_j(i) \quad \text{equal 1 when } j \text{ is prime and 0 when it is composite}$$

Proof. From the previous lemma and propositions 1 and 2: in each of this two products the factors are themselves products that equal 0 or 1, where those with value 1 stand for the relative primes (i) with the corresponding j . So only with a prime value of j -with no divisors different from 1 and j - can the total product be 1 instead of 0. There is no need to exceed the square root because divisors distinct from that root always form couples with one of the numbers smaller than it. □

So, if j is seen as a variable, both products can be seen as 'indicator functions' for the set of primes above 3. Now, according to the usual notation, the Prime counting function is $\pi(x) \equiv \#\{p \le x\}$ for real abscissas in general, where the numbers p are primes and # is cardinality.

Proposition 4. For $10 < n \in \mathbb{Z}$:

$$\pi(n) = 4 + \sum_{j=11}^{n} \prod_{i=2}^{\lfloor \sqrt{j} \rfloor} v_i(j) \tag{4.1}$$

$$\pi(n) = 4 + \sum_{j=11}^{n} \prod_{i=2}^{\lfloor \sqrt{j} \rfloor} v_j(i) \tag{4.2}$$

Proof. It is obvious from the previous proposition 3, given that there are only four primes before 11. □

Theorem 3. Let O be the set of odd integers and $10 < n \in O$:

$$\pi(n) = 4 + \sum_{\substack{j=11 \\ j \in O}}^{n} \prod_{i \in O \cap [3, \sqrt{j}]} \prod_{k=1}^{\lfloor i/2 \rfloor} \left| \sin^{-1}\left(\frac{k\,\pi}{i}\right) \sin\left(\frac{k\,\pi\, j}{i}\right) \right| \tag{4.3}$$

$$\pi(n) = 4 + \sum_{\substack{j=11 \\ j \in O}}^{n} \prod_{i \in O \cap [3, \sqrt{j}]} \prod_{k=1}^{\lfloor j/2 \rfloor} \left| \sin^{-1}\left(\frac{k\,\pi}{j}\right) \sin\left(\frac{k\,\pi\, i}{j}\right) \right| \tag{4.4}$$

Proof. It is clear using proposition 3 if we remind that only odd integers can be prime above 2 and that it is enough to check each *GCD* with smaller odd integers, from 3 until the square root, to confirm if a new odd number is prime. We have substituted the rp-waves with (2.1). □

5. Basic practical considerations

As the whole idea is clearly based on the arguments of the sines, it is also possible to design a short equivalent algorithm to evaluate the Phi function that avoids the calculation of sines and the products. For instance, with (3.1) we can evaluate directly and successively the quotients $n \cdot k / i$ in the arguments, for each i from 2 until $n - 1$ and each k from $\lfloor i / 2 \rfloor$ until 1, so as to check if they are integer using for instance the floor function -as long as the computer's roundings are reliable for numbers of the required sizes-. So not all quotients need to be evaluated because once one of them is integer it is possible to skip the next ones (with lower k) for the same i and test the ones for the next i, without adding 1 in that case (of an integer quotient) to a total sum that must begin from 1 -which is relatively prime with every other n- before evaluating quotients for $i = 2$.

This method can be optimized or combined with other more or less elementary technics to be compared with others, beginning with the direct testing of common divisors with smaller integers, although it is in principle faster to combine the Euler's formula (in the introduction) with the previous factorization of the number to obtain its prime divisors, because for any natural n we can check if it is prime doing not more divisions than its square root. Anyway the basic algorithm described is of course useful for not very big numbers.

Simple adaptations for the Prime counting function can also be done from that algorithm and section 4: without refinements the same speed inconvenient arises for big numbers, but it is not quite different from the problems related with other elementary formulas also discussed in the introduction, like Willans'.

Nevertheless, algorithms in general and of course the more sophisticated ones available today don't allow a short and direct display of an exact way to count primes or to evaluate the Totient function as the previous formulas do, as claimed previously. In this sense, maybe the more compact expressions that would arise by simply replacing (2.1) in (4.1) are also of some illustrative or didactical interest.

Acknowledgments

I wish to thank Professors Fernando Chamizo and Raúl Sánchez for the reading of a first version and some formal suggestions.

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